

On the Boundary of the Dispersion-Managed Soliton Existence¹

P. M. Lushnikov

Landau Institute for Theoretical Physics, Russian Academy of Sciences, ul. Kosygina 2, Moscow, 117990 Russia

Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico, 87545

e-mail: lushnikov@lanl.gov

Received June 19, 2000; in final form, July 12, 2000

A breathing soliton-like structure in a dispersion-managed optical fiber system is studied. It is proved that, for negative average dispersion, the breathing soliton is forbidden, provided that the modulus of average dispersion exceeds a threshold which depends on the soliton amplitude. © 2000 MAIK "Nauka/Interperiodica".

PACS numbers: 42.65.Tg; 05.45.Yv; 42.79.Sz; 42.81.Dp

Propagation of an optical pulse in nonlinear media with varying dispersion is both a fundamental [1] and an important applied problem [2–8], because the dispersion-managed (DM) system, which is a system with periodic dispersion variation along an optical fiber, is one of the most prospective candidates for ultrafast high-bit-rate optical communication lines. Lossless propagation of an optical pulse in a DM fiber is described by a nonlinear Schrödinger equation (NLS) with periodically varying dispersion $d(z)$:

$$iu_z + d(z)u_{tt} + |u|^2u = 0, \quad (1)$$

where u is the envelope of the optical pulse, z is the propagation distance, and all quantities are dimensionless. Consider a two-step periodic dispersion map $d(z) = d_0 + \tilde{d}(z)$, where $\tilde{d}(z) = d_1$ for $0 < z + nL < L_1$ and $\tilde{d}(z) = d_2$ for $L_1 < z + nL < L_1 + L_2$; d_0 is the path-averaged dispersion; d_1 and d_2 are the amplitudes of dispersion variation subject to the condition $d_1L_1 + d_2L_2 = 0$; $L \equiv L_1 + L_2$ is the dispersion compensation period; and n is an arbitrary integer number. Equation (1) also describes the pulse propagation in a fiber with losses compensated by periodically placed amplifiers, if the distance between amplifiers is much less than L .

In a linear regime, in which the nonlinear term in Eq. (1) is negligible, the periodical variation of dispersion is a way to overcome pulse broadening due to chromatic dispersion, provided that the residual dispersion d_0 is small enough. However, in a real optical fiber, the nonlinear term in Eq. (1) is important, because the optical pulse amplitude should be large enough to get a high signal/noise ratio. One of the fascinating features of the DM system is the numerical observation of a space-breathing soliton-like structure, which is called the DM soliton, for both positive and negative residual dispersion d_0 [9]. This observation is in sharp contrast

with the system described by the NLS with constant dispersion, where stable soliton propagation is possible only for positive dispersion [10], because the nonlinearity can continuously compensate the positive dispersion only. In the DM soliton, the balance between the nonlinearity and dispersion is achieved, on average, over the dispersion period L , which lifts the requirement for the positive dispersion sign. Nevertheless, it was never proved that the DM soliton really exists, because there is a possibility that this is a rather long-lived quasi-stable breathing pulse which decays in a long distance z . It is shown here that for negative d_0 a DM soliton can exist only if $|d_0|$ is small enough to allow nonlinear compensation of pulse broadening due to the dispersion over distance L .

Equation (1) can be written in the Hamiltonian form $iu_z = \delta H / \delta u^*$, where the Hamiltonian

$$H = \int \left[d(z)|u_t|^2 - \frac{|u|^4}{2} \right] dt \quad (2)$$

is an integral of motion on each interval of a constant dispersion $d(z) = \text{const}$. Equation (1) is reduced to the usual NLS on such intervals. At points $z = nL$ and $z = nL + L_1$, where n is an arbitrary integer number, the Hamiltonian experiences jumps due to jumps of the dispersion, although the value of u is a continuous function of z at these points. In contrast to the Hamiltonian, the time-averaged optical power $N = \int |u|^2 dt$, or number of particles in the quantum mechanical interpretation of NLS (in this interpretation, the coordinate z means some "time" and actual time t has the meaning of "coordinate") is an integral of motion for all z . Consider the z dependence of the quantity $A = \int t^2 |u|^2 dt$. A/N is the average width of a time distribution of u , or simply $\langle t^2 \rangle$ in the quantum mechanical interpretation of NLS.

¹ This article was submitted by the authors in English.

Using (1) and integrating by parts, one gets for the first z derivative

$$A_z = d(z) \int 2it(uu_t^* - u^*u_t) dt. \quad (3)$$

In a similar way after the second differentiation with respect to z , one gets

$$A_{zz} = 4dH + 4d^2X + \frac{d_z}{d}A_z, \quad (4)$$

where $X \equiv \int |u_t|^2 dt$. It follows from Eq. (3), which is often called the virial theorem (see, e.g., [11, 12]), that A_z experiences finite jumps corresponding to jumps of a step-wise function $d(z)$:

$$\begin{aligned} A_z|_{z=L_1+0} &= \frac{d_0+d_2}{d_0+d_1} A_z|_{z=L_1-0}, \\ A_z|_{z=L+0} &= \frac{d_0+d_1}{d_0+d_2} A_z|_{z=L-0}. \end{aligned} \quad (5)$$

Set $X(z) = X_0 + \delta X(z)$, $X(0) \equiv X_0$; then one can integrate Eq. (4) over the intervals $(0, L_1)$, (L_1, L) :

$$\begin{aligned} A_z|_{z=L_1-0} &= A_z|_{z=0+0} \\ &+ 4 \int_0^{L_1} [(d_0+d_1)H_1 + (d_0+d_1)^2X] dz, \\ A_z|_{z=L-0} &= A_z|_{z=L_1+0} \\ &+ 4 \int_{L_1}^L [(d_0+d_2)H_2 + (d_0+d_2)^2X] dz, \end{aligned} \quad (6)$$

where

$$\begin{aligned} H_1 &= (d_0+d_1)X_0 - Y_0, \\ H_2 &= (d_0+d_2)X_0 - Y_0 - (d_1-d_2)\delta X|_{z=L_1} \end{aligned} \quad (7)$$

are the Hamiltonian values on the intervals $(0, L_1)$, (L_1, L) respectively:

$$Y(z) \equiv \int \frac{|u|^4}{2} dt, \quad Y_0 \equiv Y(0).$$

Here, the conservation of H_1 on interval $(0, L_1)$ is used in deriving the expression for H_2 .

The DM soliton solution of Eq. (1) (see [13]) is given by $u = \tilde{u}(z, t)\exp(ikz)$, where k is an arbitrary real constant and $\tilde{u}(z+L, t) = \tilde{u}(z, t)$ is a periodic function of z , $\tilde{u}(z, t)|_{|t| \rightarrow \infty} \rightarrow 0$. Thus, for a DM soliton $A_z|_{z=L+0} =$

$A_z|_{z=0+0}$. This condition can be recast, via Eqs. (5)–(7), into the form

$$\begin{aligned} &L(d_1+d_0) \left[2d_0X_0 - Y_0 + (d_1-d_2) \frac{L_2}{L} \delta X \right]_{z=L_1} \\ &+ \int_0^{L_1} (d_0+d_1)^2 \delta X dz + \int_{L_1}^L (d_0+d_2)^2 \delta X dz = 0. \end{aligned} \quad (8)$$

The next step is to consider the $\delta X(z)$ dependence. Using Eq. (1) and integrating by parts, one can get

$$X_z = 4 \int \phi_t R_t R^3 dt, \quad (9)$$

where $u \equiv R e^{i\phi}$, ϕ and R are real, and $R \geq 0$. Consider an upper bound of X_z , which is given by a chain of inequalities

$$4 \int \phi_t R_t R^3 dt \leq 4 \max(R^2) \int |\phi_t R_t R| dt \leq 4 X^{3/2} N^{1/2}, \quad (10)$$

where the following inequalities are used:

$$\begin{aligned} 2\phi_t R_t R &\leq (\phi_t R)^2 + R_t^2, \\ \max(R^2) &\leq \int_{-\infty}^t |(R^2)_t| dt' \\ &\leq \int |(R^2)_t| dt \leq 2 \int R |R_t| dt \leq 2 N^{1/2} X^{1/2} \end{aligned} \quad (11)$$

(in the last expression, the Cauchy–Schwarz inequality is also used). Equations (9) and (10) can be integrated over z to give (it is assumed below that $2X_0^{1/2} N^{1/2} \max(L_1, L_2) < 1$)

$$X \leq X_0 / (1 - 2X_0^{1/2} N^{1/2} z)^2. \quad (12)$$

In a similar way, using inequality $X_z \geq -4 \int |\phi_t R_t| R^3 dt$ following from Eq. (9), one can get the lower bound for $X(z)$:

$$X \geq X_0 / (1 + 2X_0^{1/2} N^{1/2} z)^2. \quad (13)$$

For the DM soliton $X(L) = X_0$, and, thus, it is more convenient to use similar inequalities for $L_1 < z < L$:

$$\begin{aligned} &\frac{X_0}{(1 + 2X_0^{1/2} N^{1/2} (L-z))^2} \\ &\leq X \leq \frac{X_0}{(1 - 2X_0^{1/2} N^{1/2} (L-z))^2}. \end{aligned} \quad (14)$$

Equations (8), (12)–(14) result in the inequality

$$|2d_0X_0 - Y_0| \leq \frac{|d_1 - d_2|L_2X_0}{L} \left[\frac{1}{(1 - 2X_0^{1/2}N^{1/2}L_1)^2} - 1 \right] + \frac{2X_0^{3/2}N^{1/2}}{|d_0 + d_1|L} \left[\frac{(d_0 + d_1)^2L_1^2}{1 - 2X_0^{1/2}N^{1/2}L_1} + \frac{(d_0 + d_2)^2L_2^2}{1 - 2X_0^{1/2}N^{1/2}L_2} \right]. \quad (15)$$

Equation (15) is the main result of this paper. Equation (15) is a consequence of the initial assumption that the DM soliton exists for given parameters L_1, L_2, d_0, d_1, d_2 and integral values X_0, Y_0, N , which depend on $u|_{z=0}$ only. Thus, the DM soliton can exist only if this inequality is fulfilled.

Note that if one assumes uniqueness of the DM soliton solution for a given k and soliton width, then, as shown in [13], $|u|_{z=0} = |u|_{z=L_1}$. In such a case, the term $\delta X|_{z=L_1}$ in Eq. (8) vanishes and instead of (15), one can get a more strict inequality. However, this possibility is disregarded here for the sake of generality.

To clarify the physical consequences of Eq. (15), consider an optical pulse with a typical amplitude p and a typical time width t_0 . Then, $N \sim |p|^2 t_0$, $X_0 \sim |p|^2 / t_0$ and, thus, $X_0^{1/2} N^{1/2} L \sim L/Z_{nl}$, where $Z_{nl} = 1/|p|^2$ is the characteristic nonlinear length. In typical experimental conditions, the nonlinearity is small: $L/Z_{nl} \ll 1$, and the denominators in (15) can be expanded in series to give

$$|2d_0X_0 - Y_0| \leq \frac{2X_0^{3/2}N^{1/2}}{L} \times \left[2|d_1 - d_2|L_1L_2 + |d_0 + d_1|L_1^2 + \frac{(d_0 + d_2)^2L_2^2}{|d_0 + d_1|} \right] + O\left(\frac{d_1L^3}{t_0Z_{nl}^3}\right). \quad (16)$$

Provided that d_0 is negative, both terms on the left-hand side of Eq. (16) have the same sign and, thus, the right-hand side should be greater than, or equal to, $2|d_0|X_0 + Y_0$. Assuming $d_1 \gg |d_0|$, one can get from Eq. (16) the following estimate ($Y_0 \sim t_0/Z_{nl}^2$):

$$\frac{2|d_0|}{t_0Z_{nl}} + \frac{t_0}{Z_{nl}^2} \lesssim \frac{4L_1d_1}{Z_{nl}^2t_0} \left(1 + \frac{L_1}{L} \right). \quad (17)$$

Consider the strong dispersion management limit $Z_{disp}/L \ll 1$, where $Z_{disp} \equiv t_0^2/d_1$ is the typical dispersion length. This limit implies that the optical pulse experiences strong oscillation at each period L due to the dispersion. Then Eq. (17) reduces to

$$-\frac{d_0}{d_1} \lesssim \frac{6L_1}{Z_{nl}} \left(1 + \frac{L_1}{L} \right), \quad (18)$$

i.e., a nonlinearity (amplitude of the optical pulse) should be strong enough to allow the DM soliton solution to exist for a given negative d_0 .

Equation (15) gives the necessary, but not sufficient, condition for the existence of the DM soliton. In other words, the violation of inequality (15) means that the DM soliton is forbidden. Of course, it would be interesting to find to what extent this necessary condition for existence is close to the sufficient one. In general, this could be done only if one found the DM soliton analytically.

Here, one can only mention that there is a qualitative correspondence between the threshold of DM soliton existence, following from the analytical condition (15) and from the numerical investigation of the DM soliton. Namely, the maximum value of $|d_0|$ ($d_0 < 0$) for which the DM soliton exists grows with an increase in the dispersion map strength L/Z_{disp} , according to both numerics (see, e.g., [14, 15]) and analytical condition (16). It also follows from Eq. (18) that for an asymmetric dispersion map $L_1 \neq L_2$ the maximum possible value of $|d_0|$ grows as L_1 increases (for fixed L, Z_{nl}, d_1), in correspondence with Fig. 3 in [15].

Equation (15) also has a clear physical meaning in another limit $d_0/d_1 \gg L/Z_{nl}$, $Z_{disp} \gg L$, and $Z_{nl} \gg L$, in which Eq. (15) reduces to

$$(2d_0X_0 - Y_0)/Y_0 = O(L/Z_{disp}) \ll 1. \quad (19)$$

Equality $2d_0X_0 = Y_0$ exactly corresponds to the one-soliton solution of the NLS with dispersion d_0 (see [10]), where the dispersion d_0 and the nonlinearity continuously balance each other. Thus, in the limit $Z_{disp} \gg L$, which is called a weak dispersion limit, we recover the usual NLS describing the path-averaged (over the space period L) DM soliton dynamics, provided d_0 is large enough. A weak dispersion management limit was studied earlier [1, 16–18]. Note that an additional condition $d_0/d_1 \gg L/Z_{nl}$ allows the amplitude d_1 of the dispersion variation to be much higher still than d_0 , because one assumes $L \ll Z_{nl}$.

To summarize, the necessary analytical condition (15) for the existence of the DM soliton is established. From the physical point of view, this condition means that the DM soliton solution can exist only if the nonlinearity is strong enough to compensate the pulse broadening due to the negative value of the average dispersion d_0 . Note that estimates in Eqs. (16)–(19) are only given here for a physical interpretation of the analytical condition (15). So far, the DM soliton solution has been obtained numerically [3, 4, 14] and by the variational [5] and other perturbative approaches [19–21]. These results are in agreement with condition (15). But analytical proof of the existence of the DM soliton in the parameter region satisfying condition (15), i.e. the sufficient condition for existence, is still an open question.

The author thanks I.R. Gabitov for helpful discussions.

Support was provided by the US Department of Energy, under contract W-7405-ENG-36, RFBR and the program of the Russian Government Support for Leading Scientific Schools.

REFERENCES

1. V. E. Zakharov, in *Optical Solitons: Theoretical Challenges and Industrial Perspectives*, Ed. by V. E. Zakharov and S. Wabnitz (Springer-Verlag, Berlin, 1999), p. 73; V. E. Zakharov and S. V. Manakov, *Pis'ma Zh. Éksp. Teor. Fiz.* **70**, 573 (1999) [*JETP Lett.* **70**, 578 (1999)].
2. C. Lin, H. Kogelnik, and L. G. Cohen, *Opt. Lett.* **5**, 476 (1980).
3. M. Nakazawa and H. Kubota, *Electron. Lett.* **31**, 216 (1995).
4. N. J. Smith, F. M. Knox, N. J. Doran, *et al.*, *Electron. Lett.* **32**, 54 (1996).
5. I. Gabitov and S. K. Turitsyn, *Opt. Lett.* **21**, 327 (1996); *Pis'ma Zh. Éksp. Teor. Fiz.* **63**, 814 (1996) [*JETP Lett.* **63**, 861 (1996)].
6. S. Kumar and A. Hasegawa, *Opt. Lett.* **22**, 372 (1997).
7. P. V. Mamyshev and N. A. Mamysheva, *Opt. Lett.* **24**, 1454 (1999).
8. L. F. Mollenauer, P. V. Mamyshev, J. Gripp, *et al.*, *Opt. Lett.* **25**, 704 (2000).
9. J. H. B. Nijhof, N. J. Doran, W. Forysiak, and F. M. Knox, *Electron. Lett.* **33**, 1726 (1997).
10. V. E. Zakharov and A. B. Shabat, *Zh. Éksp. Teor. Fiz.* **61**, 118 (1971) [*Sov. Phys. JETP* **34**, 62 (1972)].
11. V. E. Zakharov, *Zh. Éksp. Teor. Fiz.* **62**, 1745 (1972) [*Sov. Phys. JETP* **35**, 908 (1972)].
12. P. M. Lushnikov, *Pis'ma Zh. Éksp. Teor. Fiz.* **62**, 447 (1995) [*JETP Lett.* **62**, 461 (1995)].
13. S. K. Turitsyn, J. H. B. Nijhof, V. K. Mezentsev, and N. J. Doran, *Opt. Lett.* **24**, 1871 (1999).
14. A. Berntson, N. J. Doran, W. Forysiak, and J. H. B. Nijhof, *Opt. Lett.* **23**, 900 (1998).
15. A. Berntson, D. Anderson, N. J. Doran, *et al.*, *Electron. Lett.* **34**, 2054 (1998).
16. A. Hasegawa and Y. Kodama, *Solitons in Optical Communications* (Oxford Univ. Press, New York, 1995).
17. Yu. L. Lvov and I. R. Gabitov, *chao-dyn/9907007* (1999).
18. S. B. Medvedev and S. K. Turitsyn, *Pis'ma Zh. Éksp. Teor. Fiz.* **69**, 465 (1999) [*JETP Lett.* **69**, 499 (1999)].
19. S. K. Turitsyn and V. K. Mezentsev, *Pis'ma Zh. Éksp. Teor. Fiz.* **67**, 616 (1998) [*JETP Lett.* **67**, 640 (1998)]; S. K. Turitsyn, *Phys. Rev. E* **58**, 1256 (1998).
20. T. Lakoba and D. J. Kaup, *Electron. Lett.* **34**, 1124 (1998).
21. P. M. Lushnikov, *Opt. Lett.* **25**, 1144 (2000) (in press).